1 WS5 1

For certain numerical values of c, the answer when we evaluate the integral $\int \frac{1}{x^2+6x+c}dx$ involves an arctangent. By completing the square in the denominator of the integrand, determine those values of c, and evaluate the corresponding integral for an arbitrary such value of c.

Note that to complete the square of a quadratic expression ax^2+bx+c, first factor out a: $a(x^2+\frac{b}{a}x+\frac{c}{a})$. Then, in the parentheses, add and subtract $(\frac{b}{2a})^2$ (this adds 0; adding 0 doesn't change the expression):

$$a(x^2+\frac{b}{a}x+\frac{c}{a}) = a(x^2+\frac{b}{a}x+\frac{c}{a}\pm(\frac{b}{2a})^2) = a(x^2+\frac{b}{a}x+(\frac{b}{2a})^2+\frac{c}{a}-(\frac{b}{2a})^2) \quad (1)$$

$$= a((x+\frac{b}{2a})^2+\frac{c}{a}-(\frac{b}{2a})^2) \quad (2)$$

For us, completing the square of x^2+6x+c amounts to adding and subtracting $(\frac{b}{2a})^2 = (\frac{6}{2\cdot 1})^2 = 3^2 = 9$ to the expression: $x^2+6x+c = x^2+6x+c\pm 9 = (x+3)^2+c-9$.

Thus, $\int \frac{1}{x^2+6x+c}dx = \int \frac{1}{(x+3)^2+c-9}dx = \int \frac{1}{u^2+c-9}du$ for $u = x+3$, $du = dx$. Since $\frac{1}{a}\tan^{-1}(\frac{x}{a})+C = \int \frac{1}{x^2+a^2}dx$, we need $c-9$ to be positive since x^2+a^2 is always x^2 plus a positive quantity. Thus $c-9>0$, so $c>9$.

Then, for such a value of c, $\sqrt{c-9}$ is a in the formula. Thus $\int \frac{1}{u^2+c-9}du = \frac{1}{\sqrt{c-9}}\tan^{-1}(\frac{u}{\sqrt{c-9}})+C = \frac{1}{\sqrt{c-9}}\tan^{-1}(\frac{x+3}{\sqrt{c-9}})+C$.

2 WS5 2

Let g be a differentiable function defined on $[0,1]$, with $|g(t)|<3$ for $0\le t\le 1$. Thus $16-(g(t))^2$ is strictly positive on $[0,1]$.

(a) Substitute $u=g(t)$, and then evaluate the integral $\int \frac{g'(t)}{\sqrt{16-(g(t))^2}}dt$.

(b) Suppose $g(0)=0$. Find a value $g(1)$ so that $\int_0^1 \frac{g'(t)}{\sqrt{16-(g(t))^2}}dt = \frac{\pi}{3}$.

(a) Substituting $u=g(t)$ has $du=g'(t)dt$. Then $\int \frac{g'(t)}{\sqrt{16-(g(t))^2}}dt = \int \frac{du}{\sqrt{16-u^2}}$. Since

$$\sin^{-1}(\tfrac{x}{a})+C = \int \tfrac{1}{\sqrt{a^2-x^2}}dx$$

(note there's no $\frac{1}{a}$ in front of $\sin^{-1}$, like with $\tan^{-1}$) we have $\int \frac{du}{\sqrt{16-u^2}} = \sin^{-1}(\frac{u}{4})+C = \sin^{-1}(\frac{g(t)}{4})+C$.

(b) If $g(0)=0$, then

$$\int_0^1 \tfrac{g'(t)}{\sqrt{16-(g(t))^2}}dt = \sin^{-1}(\tfrac{g(1)}{4})-\sin^{-1}(\tfrac{g(0)}{4}) = \sin^{-1}(\tfrac{g(1)}{4})-\sin^{-1}(0) = \sin^{-1}(\tfrac{g(1)}{4}) = \tfrac{\pi}{3}$$

by the Fundamental Theorem of Calculus. Then, applying sin to both sides, of the equation $\sin^{-1}(\frac{g(1)}{4}) = \frac{\pi}{3}$ we have $\sin\frac{\pi}{3} = \frac{g(1)}{4}$, so

$$g(1) = 4\cdot\sin(\tfrac{\pi}{3}) = 4\cdot\tfrac{\sqrt{3}}{2} = 2\sqrt{3}.$$

3 WS5 3

Jo drops a marble from her apartment window, 20 meters above the ground. At the same height but 12 meters away, Doe watches the marble fall. When the height of the marble above the ground is h, let θ be the angle between L and M, where L is the horizontal line joining Jo and Doe, and where M represents Doe's line of sight to the marble.

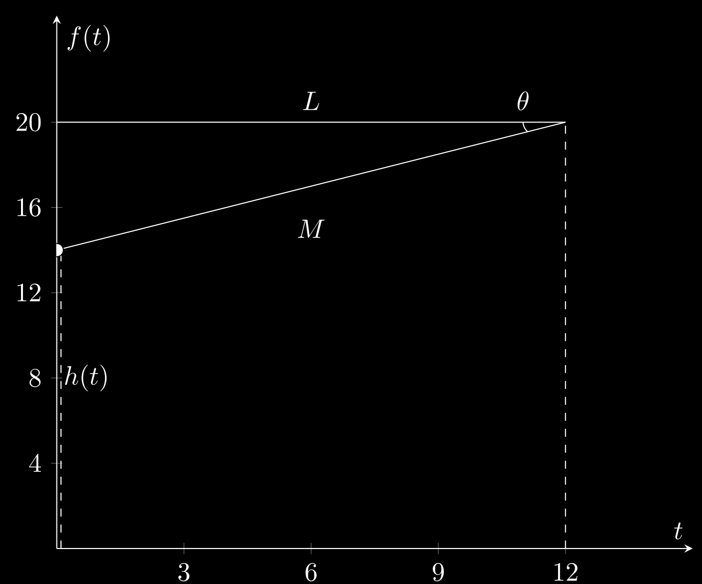
(a) Draw a picture of the scene, including L, M, and θ. Put the horizontal axis at ground level, and indicate the position of the marble $h(t)$ meters above ground level at time t.

(b) Find θ in terms of h, and then calculate $d\theta/dh$.

(c) Assume that $h(t) = 20 - 4.9t^2$ until the marble hits the ground. Find $d\theta/dt$. Show geometrically why $d\theta/dt > 0$, and then tell why the formula for $d\theta/dt$ tells us that $d\theta/dt > 0$.

Note: It turns out that there is t_0 for which $d\theta/dt$ has a maximum value. For $\theta(t_0)$ the marble appears to be falling the fastest. Can you find the value of t_0?

(a)



(b) Note that in the triangle formed by L and M, the side opposite θ has length $20 - h(t)$, and L, the adjacent side, has length 12.

Thus $\tan(\theta) = \frac{20-h(t)}{12}$, so $\theta = \tan^{-1}(\frac{20-h(t)}{12})$.

Now $d\theta/dh = \frac{d}{dh}(\tan^{-1}(\frac{20-h(t)}{12})) = \frac{1}{1+(\frac{20-h(t)}{12})^2} \cdot \frac{-1}{12} = \frac{-1}{12+\frac{(20-h(t))^2}{12}} = \frac{-12}{144+(20-h(t))^2}$ by the chain rule.

(c) If $h(t) = 20 - 4.9t^2$, then $dh/dt = -9.8t$. By the chain rule, $\frac{d\theta}{dt} = \frac{d\theta}{dh} \cdot \frac{dh}{dt}$. Thus

$$d\theta/dt = \frac{-12}{144+(20-h(t))^2} \cdot -9.8t = \frac{117.6t}{144+(20-h(t))^2} = \frac{117.6t}{144+(4.9t^2)^2} = \frac{117.6t}{144+24.01t^4}.$$

Geometrically, $d\theta/dt > 0$ since the angle θ increases in degree as the marble falls.

Also, the formula tells us that $d\theta/dt > 0$: since $t > 0$, the formula $d\theta/dt = \frac{117.6t}{144+24.01t^4}$ stays positive.

To find the value of t_0, we will do the first derivative test on $d\theta/dt$ (i.e. find the critical points, or the ts where $\frac{d}{dt}(d\theta/dt) = 0$.

$$\frac{d}{dt}(d\theta/dt) = \frac{d}{dt}\frac{117.6t}{144+24.01t^4} = \frac{117.6\cdot(144+24.01t^4)-117.6t\cdot 96.04t^3}{(144+24.01t^4)^2}.$$

This fraction will only be zero when the top is zero, we expand it here:

$117.6 \cdot (144 + 24.01t^4) - 117..6t \cdot 96.04t^3 = 16,934.4 + 2,823.576t^4 - 11,294.304t^4 = 16,934 - 8,470.728t^4 = 0.$

Thus $t = (\frac{16,934}{8,470.728})^{\frac{1}{4}} \approx 1.2\ s$